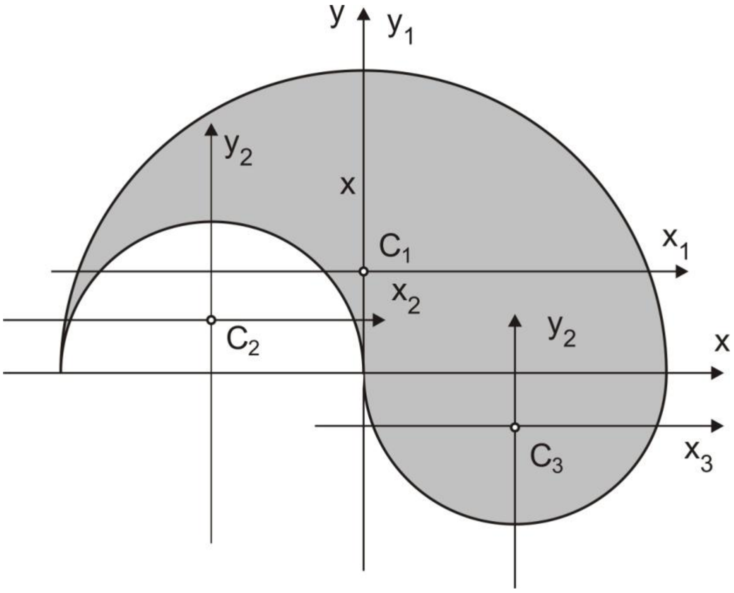
Zadatak 5.

Za datu sliku, R=4cm, odrediti moment inercije i glavne momente inercije kao i poluprečnike elipse inercije.



$A_1 = \frac{(2R)^2\pi}{2} = 2R^2\pi = 2\cdot 4^2\pi = 32\pi = 100.53\ cm^2 \quad C_1\left(0, \frac{4\cdot 2R}{3\pi}\right) = (0; 3.395)$

$A_2 = \frac{R^2\pi}{2} = 25.133 cm^2, \quad C_2\left(-R; \frac{4R}{3\pi}\right) = (-4; 1.697)$

$A_3 = \frac{R^2\pi}{2} = 25.133 cm^2, \quad C_2\left(R; -\frac{4R}{3\pi}\right) = (4; -1.697)$

$A = \sum A = A_1 - A_2 + A_3 = 2R^2\pi - \frac{R^2\pi}{2} + -\frac{R^2\pi}{2} = 2R^2\pi = 100.53 cm^2$

$S_x = \sum A_i \cdot y_i = A_1\cdot y_1 - A_2\cdot y_2 + A_3\cdot y_3 = 2R^2\pi\cdot\frac{4\cdot 2R}{3\pi} - \frac{R^2\pi}{2}\cdot\frac{4\cdot 2R}{3\pi} + \frac{R^2\pi}{2}\cdot\left(-\frac{4\cdot 2R}{3\pi}\right)$

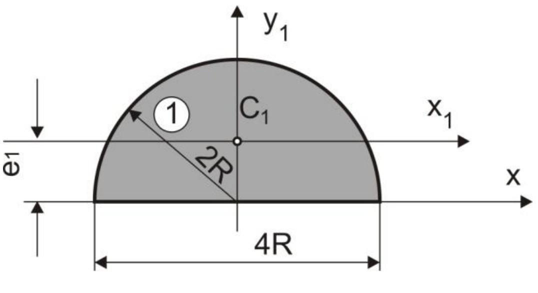
$S_x = \frac{16R^3}{3} - \frac{4R^3}{3} = 4R^3 = 256\ cm^3$

$S_y = \sum A_i\cdot x_i = A_1\cdot x_1 - A_2\cdot x_2 + A_3\cdot x_3 = 2R^2\pi\cdot 0 - \frac{R^2\pi}{2}\cdot(-R) + \frac{R^2\pi}{2}\cdot R = R^3\pi$

$S_y = R^3\pi = 256\pi = 804.247 cm^3$

$x_C = \frac{S_y}{A} = \frac{R^3\pi}{2R^2\pi} = \frac{R}{2} = \frac{4}{2} = 2 cm^2$

$y_C = \frac{S_x}{A} = \frac{4R^3}{2R^2\pi} = \frac{256}{100.53} = 2.546 cm^2$



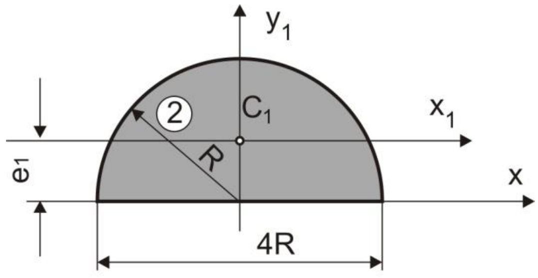
$e_1 = \frac{4\cdot 2R}{3\pi} = \frac{8R}{3\pi} = \frac{8\cdot 4}{3\pi} = 3.395$

$I_{y1} = \frac{(2R)^4\pi}{8} = \frac{8^4\pi}{8} = \frac{4096\ \pi}{8} = 512\pi = 1608.49\ cm^4$

$I_{x1} = (2R)^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 8^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 449.74\ cm^4$

$I_x = \frac{(2R)^4\pi}{8} = \frac{8^4\pi}{8} = \frac{4096\ \pi}{8} = 512\pi = 1608.49\ cm^4$

$I_{x1y1} = 0$



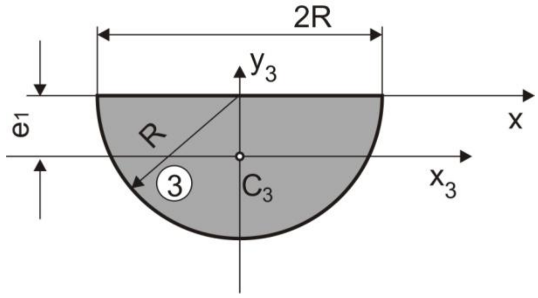
$e_1 = \frac{4R}{3\pi} = \frac{4\cdot 4}{3\pi} = 1.697\ cm$

$I_{y2} = \frac{R^4\pi}{8} = \frac{4^4\pi}{8} = \frac{256\ \pi}{8} = 100.53\ \ cm^4$

$I_{x2} = R^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 4^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 28.188\ \ cm^4$

$I_x = \frac{R^4\pi}{8} = \frac{4^4\pi}{8} = \frac{256\ \pi}{8} = 100.53\ \ cm^4$

$I_{x2y2} = 0$



$e_1 = \frac{4R}{3\pi} = \frac{4\cdot 4}{3\pi} = 1.697\ cm$

$I_{y3} = \frac{R^4\pi}{8} = \frac{4^4\pi}{8} = \frac{256\ \pi}{8} = 100.53\ \ cm^4$

$I_{x3} = R^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 4^4\left(\frac{\pi}{8} - \frac{8}{9\pi}\right) = 28.188\ \ cm^4$

$I_x = \frac{R^4\pi}{8} = \frac{4^4\pi}{8} = \frac{256\ \pi}{8} = 100.53\ \ cm^4$

$I_{x2y2} = 0$

$I_x = I_{x1} - I_{x2} + I_{x3} = 512\pi - \frac{256\ \pi}{8} + \frac{256\ \pi}{8} = I_{x1} = 512\pi = 1608.49 cm^4$

$I_y = I_{y1} - \left(I_{y2} + R^2A_1\right) + \left(I_{y3} + R^2A_3\right) = I_{y1} = 512\pi = 1608.49 cm^4$

$I_{xy} = I_{x1y1} - \left(I_{x2y2} + (-R)\cdot\frac{4R}{3\pi}\cdot A_1\right) + \left(I_{x3y3} + R\cdot\left(-\frac{4R}{3\pi}\right)\cdot A_3\right)$

$I_{xy} = 0 - \left(0 + (-R)\cdot\frac{4R}{3\pi}\cdot A_1\right) + \left(0 + R\cdot\left(-\frac{4R}{3\pi}\right)\cdot A_3\right) = 0$

Za težišne ose udaljene od x ose za y_c odnosno od y ose za x_c pomoću Štajnerove teoreme izračunavajaju se sopstveni momenti za težišne ose

$I_\xi = I_x - {y_c}^2A = 1608.49 - (2.546)^2\cdot 100.53 = 956.840 cm^4$

$I_\eta = I_y - {x_c}^2A = 1608.49 - (2)^2\cdot 100.53 = 1206.37 cm^4$

$I_{\xi\eta} = I_{xy} - x_c y_c A = 0 - 2\cdot 2.546\cdot 100.53 = -511.904 cm^4$

$\tan 2\alpha = \frac{-2I_{\xi\eta}}{I_\xi - I_\eta} = \frac{-2(-511.904)}{956.84-1206.37} = -4.1029455 \rightarrow 2\alpha = -76.3025° \rightarrow \alpha = 38.1512°$

$I_{12} = \frac{1}{2}\left(I_\xi + I_\eta\right) \pm \frac{1}{2}\sqrt{\left(I_\xi - I_\xi\right)^2 + 4I_{\xi\eta}^2} =$

$I_{12} = \frac{1}{2}(956.84 + 1206.37) \pm \frac{1}{2}\sqrt{(956.84 - 1206.37)^2 + 4\cdot 511.904^2} =$

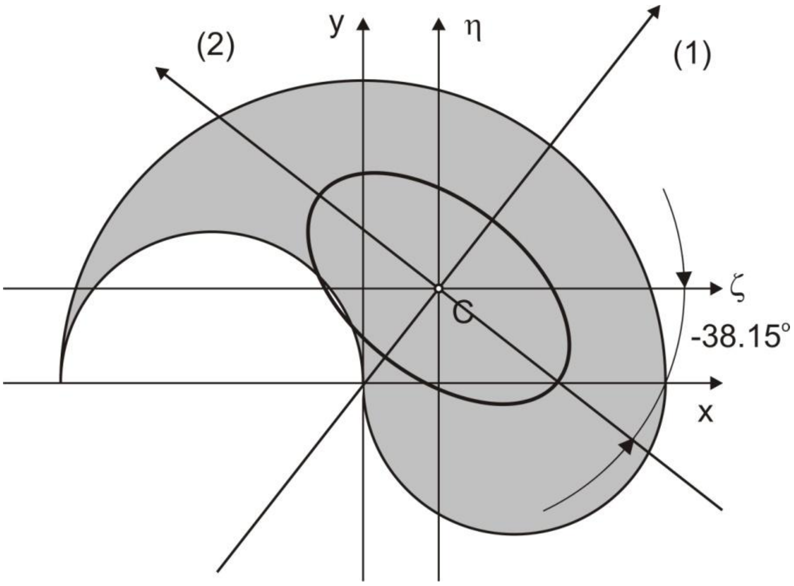
$I_{12} = 1081.605 \pm 526.888$

$I_1 = 1608.494\ cm^4$

$I_2 = 554.716\ cm^4$

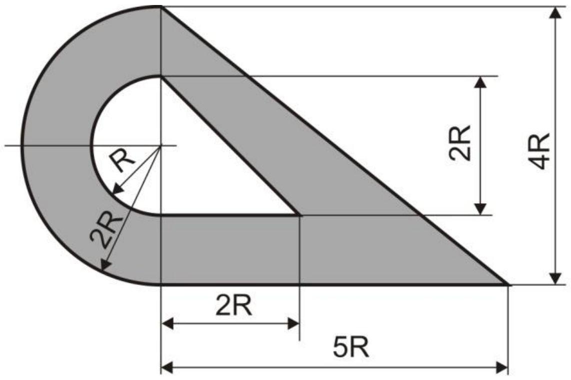
$i_1 = \sqrt{\frac{I_1}{A}} = \sqrt{\frac{1608.494}{100.531}} = \sqrt{15.999} = 4\ cm$

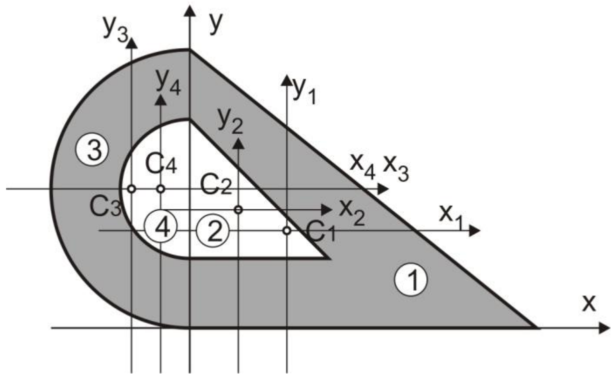
$$i_2 = \sqrt{\frac{I_2}{A}} = \sqrt{\frac{554.716}{100.531}} = \sqrt{5.5225} = 2.35\ cm$$



Zadatak 6.

Za datu sliku, R=2cm, odrediti moment inercije i glavne momente inercije kao i poluprečnike elipse inercije.





$A_1 = \frac{5R \cdot 4R}{2} = 10R^2 = 40cm^2,$

$C_1\left(\frac{5R}{3}; \frac{4R}{3}\right) = (3.333; 2.667)$

$A_2 = \frac{2R \cdot 2R}{2} = 2R^2 = 8cm^2,$

$C_2\left(\frac{2R}{3}; \frac{5R}{3}\right) = (1.333; 3.333)$

$A_3 = \frac{(2R)^2 \cdot \pi}{2} = 2R^2\pi = 25.133cm^2,$ $\quad C_3\left(-\frac{4 \cdot 2R}{3\pi}; 2R\right) = (-1.697; 4)$

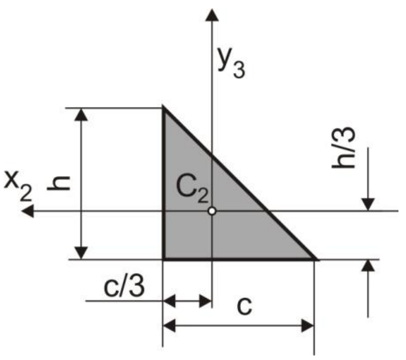
$A_4 = \frac{R^2 \cdot \pi}{2} = 6.283\ cm^2,$ $\quad C_4\left(-\frac{4 \cdot R}{3\pi}; 2R\right) = (-0.848; 4)$

$$A = \sum A = A_1 - A_2 + A_3 - A_4 = 40 - 8 + 25.137 - 6.283 = 50.854\ cm^2$$

$$x_c = \frac{\sum A_i x_i}{A} = \frac{A_1x_1 - A_2x_2 + A_3x_3 - A_4x_4}{A}$$

$$x_c = \frac{40 \cdot 3.333 - 8 \cdot 1.333 + 25.133 \cdot (-1.697) - 6.283 \cdot (-0.848)}{50.854} = \frac{90.662}{50.854} = 1.678\ cm$$

$$y_c = \frac{\sum A_i y_i}{A} = \frac{A_1y_1 - A_2y_2 + A_3y_3 - A_4y_4}{A}\ y_c = \frac{40 \cdot 2.666 - 8 \cdot 3.333 + 25.137 \cdot 4 - 6.283 \cdot 4}{50.854} = \frac{155.392}{50.854} = 3.056\ cm$$



Za trouglove 1 i 2 sopstveni momenti inercije

$I_x = \frac{ch^3}{36}$

$I_y = \frac{c^3h}{36}$

$I_{xy} = \pm \frac{c^2h^2}{72}$

Za trougao 1 sopstveni momenti inercije

$c = 5R = 5 \cdot 2 = 10\ cm$

$h = 4R = 4 \cdot 2 = 8cm$

$$I_{x1} = \frac{ch^3}{36} = \frac{5R \cdot (4R)^3}{36} = \frac{320R^4}{36} = \frac{10 \cdot 8^3}{36} = 142.222cm^4$$

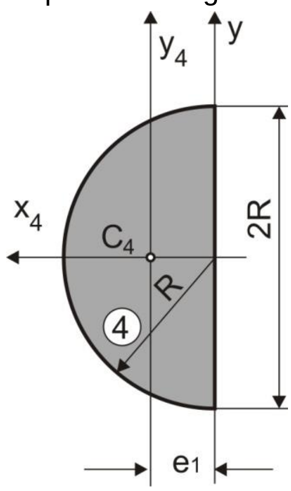
$I_{y1}=\frac{c^3h}{36}=\frac{(5R)^3\cdot 4R}{36}=\frac{10^3\cdot 8}{36}=222.222cm^4$

$I_{x1y1}=\frac{c^2h^2}{72}=\frac{(5R)^2\cdot(4R)^2}{72}=\frac{10^2\cdot 8^2}{72}=88.888cm^4$

Za trougao 2 sopstveni momenti inercije

$c=2R=2\cdot 2=4\ cm$
$h=2R=2\cdot 2=4cm$
$I_{x2}=\frac{ch^3}{36}=\frac{2R\cdot(2R)^3}{36}=\frac{4\cdot 4^3}{36}=7.111cm^4$
$I_{y2}=\frac{c^3h}{36}=\frac{(2R)^3\cdot 2R}{36}=\frac{4^3\cdot 4}{36}=7.111cm^4$
$I_{x2y2}=\frac{c^2h^2}{72}=\frac{(2R)^2\cdot(2R)^2}{72}=\frac{4^2\cdot 4^2}{72}=3.555cm^4$

Za polovine krugova 3 i 4 sopstveni momenti inercije



$e_1=\frac{4R}{3\pi}$
$I_{x4}=\frac{R^4\pi}{8}=6.283cm^4$
$I_{y4}=R^4\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)=0.1098R^4$
$I_{x4y4}=0$

Za polovinu kruga 3 $R_3=2R=2\cdot 2=4cm$

$e_1=\frac{4\cdot 2R}{3\pi}=\frac{4\cdot 4}{3\pi}=1.697cm$
$I_{x3}=\frac{R_3^4\pi}{8}=\frac{(2R)^4\pi}{8}=\frac{4^4\pi}{8}=100.531cm^4$
$I_{y3}=R_3^4\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)=(2R)^4\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)=0.1098\cdot(2R)^4=28.108cm^4$

$I_{x3y3}=0$

Za pololovinu kruga 4 $R_4=R=2cm$

$e_1=\frac{4\cdot R}{3\pi}=\frac{4\cdot 2}{3\pi}=0.848cm$
$e_1=\frac{4\cdot R}{3\pi}=\frac{4\cdot 2}{3\pi}=0.848\ cm$
$I_{x4}=\frac{R_4^4\pi}{8}=\frac{R^4\pi}{8}=\frac{2^4\pi}{8}=6.2831cm^4$
$I_{y4}=R_4^4\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)=R^4\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)=0.1098\cdot R^4=1.7568cm^4$

$I_{x3y3}=0$
$I_x=I_{x1}+y_1^2\cdot A_1-(I_{x2}+y_2^2\cdot A_2)+I_{x3}+y_3^2\cdot A_3-(I_{x4}+y_4^2\cdot A_4)$
$I_x=\left[\frac{320}{12}+\frac{16}{9}10-\frac{16}{36}-\frac{25}{9}2+16\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)+2\pi\cdot 4-\left(\frac{\pi}{8}-\frac{8}{9\pi}\right)-\frac{4\pi}{2}\right]R^4=$
$=142.22+2.666^2\cdot 40-(7.11+3.333^2\cdot 8)+100.53+4^2\cdot 25.137-(6.283+4^2\cdot 6.283)=$
$I_x=726.507cm^4$
$I_y=I_{y1}+x_1^2\cdot A_1-\left(I_{y2}+x_2^2\cdot A_2\right)+I_{y3}+x_3^2\cdot A_3-(I_4+x_4^2\cdot A_4)$
$=222.22+3.333^2\cdot 40-(7.111+1.333^2\cdot 8)+28.108+1.697^2\cdot 25.137-$
$-(1.7568+0.848^2\cdot 6.283)=$
$I_y=739.581cm^4$
$I_{xy}=I_{x1y1}+x_1y_1\cdot A_1-\left(I_{x2y2}+x_2y_2\cdot A_2\right)+I_{x3y3}+x_3y_3\cdot A_3-\left(I_{x4y4}+x_4y_4\cdot A_4\right)$

$= 88.888 + 3.333 \cdot 2.67 \cdot 40 - (3.556 + 1.333 \cdot 3.333 \cdot 8) + 0$
$+(-1.697) \cdot 4 \cdot 25.133 - (0 + (-0.848) \cdot 4 \cdot 6.283) = 255.13 cm^4$

$I_\zeta = I_x - \eta_1^2 \cdot A = 726.507 - 3.056^{\,2} \cdot 50.854 = 251.574 cm^4$

$I_\eta = I_x - \zeta_1^2 \cdot A = 739.581 - 1.678^{\,2} \cdot 50.854 = 596.392 cm^4$

$I_{\zeta\eta} = I_{xy} - \zeta\eta \cdot A = 78.222 - 1.678 \cdot 3.056 \cdot 50.854 = -182.555 cm^4$

$tg2\alpha = \frac{-2I_{\zeta\eta}}{I_\zeta - I_\eta} = \frac{-2(-182.555)}{251.574-596.392} = -1.05885 \rightarrow 2\alpha = -46.637° \rightarrow \alpha = -23.318°$

$I_{12} = \frac{1}{2}\left(I_\zeta + I_\eta\right) \pm \frac{1}{2}\sqrt{\left(I_\zeta - I_\eta\right)^2 + 4I_{\zeta\eta}^2}$

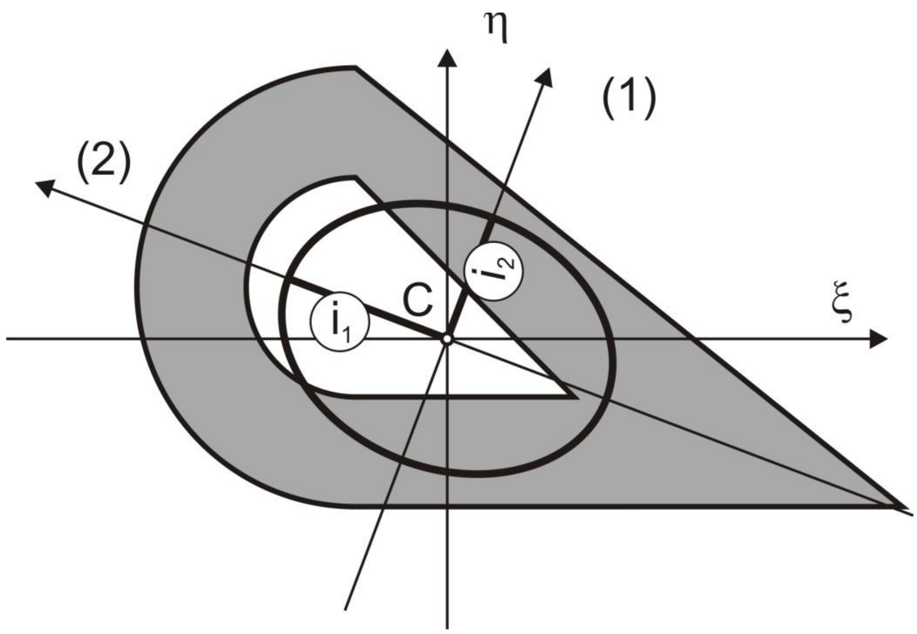
$I_{12} = \frac{1}{2}(251.574 + 596.392) \pm \frac{1}{2}\sqrt{(251.574 - 596.392)^2 + 4 \cdot 182.555^2}$

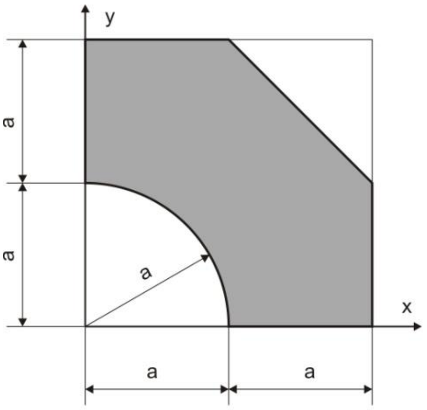
$I_{12} = 423.983 \pm 251.099$

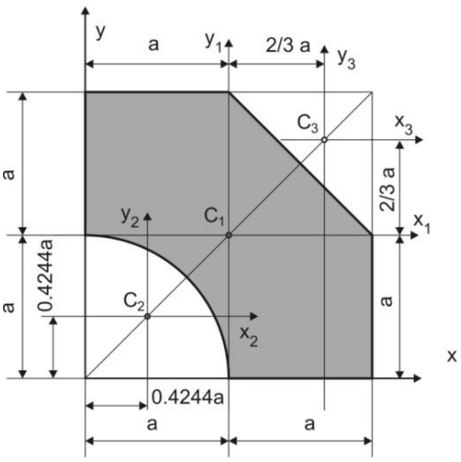
$I_1 = 423.983 + 251.099 = 675.082 \text{ cm}^4$
$I_2 = 423.983 - 251.099 = 172.884 \text{ cm}^4$

$i_1 = \sqrt{\frac{I_1}{A}} = \sqrt{\frac{675.082}{50.854}} = 3.643\ cm$

$i_2 = \sqrt{\frac{I_2}{A}} = \sqrt{\frac{172.884}{50.854}} = 1.843\ cm$







Zadatak 7.

Za datu sliku, a=1cm, odrediti moment inercije i glavne momente inercije kao i poluprečnike elipse inercije.

$A_1 = (2a)^2 = 4a^2 = 4\ cm^2 \quad C_1(a, a) = (1; 1)$

$A_2 = \frac{a^2\pi}{4} = 0.785cm^2, \quad C_2\left(\frac{4a}{3\pi}; \frac{4a}{3\pi}\right) = (0.424; 0.424)$

$A_3 = \frac{a \cdot a}{2} = 0.5cm^2, \quad C_2\left(\frac{5a}{3}; \frac{5a}{3}\right) = (1.666; 1.666)$

$A = \sum A = A_1 - A_2 - A_3 =$

$A = \sum A = 4a^2 - \frac{a^2\pi}{4} - \frac{a^2}{2} = 2.715a^2 = 2.715cm^2$

$S_x = \sum A_i \cdot y_i = A_1 \cdot y_1 - A_2 \cdot y_2 - A_3 \cdot y_3$

$S_x = 4^2 \cdot a - \frac{a^2\pi}{4}\frac{4a}{3\pi} - \frac{a^2}{2}\frac{5a}{3} = \frac{17}{6}a^3 = 2.8333\ cm^3$

$S_y = \sum A_i \cdot x_i = A_1 \cdot x_1 - A_2 \cdot x_2 - A_3 \cdot x_3$

$S_y = 4^2 \cdot a - \frac{a^2\pi}{4}\frac{4a}{3\pi} - \frac{a^2}{2}\frac{5a}{3} = \frac{17}{6}a^3 = 2.8333\ cm^3$

$x_c = \frac{S_y}{A} = \frac{2.8333\,a^3}{2.715\,a^2} = 1.0436a - 1.0436cm$

$y_c = \frac{S_x}{A} = \frac{2.8333\,a^3}{2.715\,a^2} = 1.0436a - 1.0436cm$

Iz tablica

Za kvadrat površina 1 sopstveni momenti inercije, za težišne ose

$I_{x1} = I_{y1} = \frac{(2a)^4}{12} = \frac{16a^4}{12} = 1.3333a^4 = 1.3333cm^4$

$I_{x1y1} = 0$

Za četvrtinu kruga površina 2 sopstveni momenti inercije, za težišne ose

$I_{x2} = I_{y2} = a^4\left(\frac{\pi}{16} - \frac{4}{9\pi}\right) = 0.054878a^4 = 0.05878cm^4$

$I_{x2y2} = a^4\left(\frac{4}{9\pi} - \frac{1}{8}\right) = 0.016471a^4 = 0.016471cm^4$

Za trougao površina 3 sopstveni momenti inercije, za težišne ose

$I_{x3} = \frac{c \cdot h^3}{36} = \frac{a \cdot a^3}{36} = \frac{a^4}{36} = 0.027777a^4 = 0.027777cm^2$

$I_{y3} = \frac{c^3 \cdot h}{36} = \frac{a^3 \cdot a}{36} = \frac{a^4}{36} = 0.027777a^4 = 0.027777cm^2$

$I_{x3y3} = \frac{c^2 \cdot h^2}{72} = \frac{a^2 \cdot a^2}{72} = \frac{a^4}{72} = 0.013888a^4 = 0.013888cm^2$

Za težište složene površine i ose ζiη primenom Štajnerove teoreme određuju se momenti inercije

$A_1 = 4a^2 = 4\ cm^2 \quad C_1(1.0436a - a; 1.0436a - a) = (0.0436a; 0.0436a) = (0.0436; 0.0436)$

$A_2 = \frac{a^2\pi}{4} = 0.785cm^2, \quad C_2\left(-1.0436a + \frac{4a}{3\pi}; -1.0436a + \frac{4a}{3\pi}\right) = (-0.6192; -0.6192)$

$A_3 = \frac{a \cdot a}{2} = 0.5cm^2, \quad C_2\left(\frac{5a}{3} - 1.0436a; \frac{5a}{3} - 1.0436a\right) = (0.623a; 0.623a) = (0.623; 0.623)$

$I_\zeta = I_{x1} + \eta_1^2 \cdot A_1 - (I_{x2} + \eta_2^2 \cdot A_2) - (I_{x3} + \eta_3^2 \cdot A_3) =$

$= [1.333 + 0.0436^2 \cdot 4 - (0.0549 + 0.6192^2 \cdot 0.785) - (0.02777 + 0.19406)]a^4 =$

$I_\zeta = 0.76308a^4 = 0.76308cm^4$

$I_\eta = I_{x1} + \zeta_1^2 \cdot A_1 - (I_{x2} + \zeta_2^2 \cdot A_2) - (I_{x3} + \zeta_3^2 \cdot A_3) =$

$= [1.333 + 0.0436^2 \cdot 4 - (0.0549 + 0.6192^2 \cdot 0.785) - (0.02777 + 0.623^2 \cdot 0.5)]a^4 =$

$I_\eta = 0.76308a^4 = 0.76308cm^4$

$I_{\zeta\eta} = I_{x1y1} + \zeta_1\eta_1 \cdot A_1 - (I_{x2y2} + \zeta_2\eta_2 \cdot A_3) - (I_{x3y31} + \zeta_3\eta_3 \cdot A_3) =$

$= [0 + 0.0436^2 \cdot 4 - (-0.0165 + 0.6192^2 \cdot 0.785) - (0.01389 + 0.623^2 \cdot 0.5)]a^4 =$

$I_{\zeta\eta} = -0.3241a^4 = -0.3241cm^4$

Kako je uočljiva osa simetrije pod uglom od 45°, a aksijalni momenti za ose ζiη su jednaki nije moguće odrediti tg2α, jedna glavna os je osa simetrije a druga glavna osa je upravna na nju.

$tg2\alpha = \frac{-2I_{\zeta\eta}}{I_\zeta - I_\eta} = \frac{-2(-0.3241)}{0.76308 - 0.76308}$

$I_{12} = \frac{1}{2}(I_\zeta + I_\eta) \pm \frac{1}{2}\sqrt{(I_\zeta - I_\eta)^2 + 4I_{\zeta\eta}^2}$

$I_1 = 0.76308a^4 + 0.3241a^4 = 1.0872a^4 = 1.0872cm^4$

$I_{12} = 0.76308a^4 - 0.3241a^4 = 0.43898a^4 = 0.43898cm^4$

$i_1 = \sqrt{\frac{I_1}{A}} = \sqrt{\frac{1.0872\,a^4}{2.715\,a^2}} = \sqrt{0.40044a^2} = 0.6328\,a = 0.6328\,cm$

$i_2 = \sqrt{\frac{I_2}{A}} = \sqrt{\frac{0.43898\,a^4}{2.715\,a^2}} = \sqrt{0.1616869a^2} = 0.4021a = 0.4021\,cm$